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Excitation of a Bose–Einstein condensate in a time-dependent magnetic field

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Abstract
Using an inverse scattering method, we find the exact bright and dark soliton solutions of the nonlinear Gross–Pitaevskii equation with external potentials. We also investigate the nonlinear excitation of a Bose–Einstein condensate under external fields. We reveal an efficient means of controlling solitons in a Bose–Einstein condensate for use in future experiments.

1. Introduction

The realization of Bose–Einstein condensation (BEC) of weakly interacting atomic gases has strongly stimulated the exploration of the nonlinear properties of matter waves [1–3]. Of particular interest are macroscopically excited Bose condensed states, such as solitons, nonlinear effects and vortices [4–6]. Conceptually, as a kind of macroscopic quasiparticle, solitons provide a link from BEC physics to fluid mechanics, nonlinear optics and fundamental particle physics. Bright and dark solitons in Bose–Einstein condensates correspond to attractive and repulsive interactions (s-wave scattering length $a < 0$ or $a > 0$). Recently the theoretical and experimental studies have focused on the dynamics and stability of dark solitons of two-component Bose–Einstein condensates. In particular, it is known that the Gross–Pitaevskii equation which describes the condensate in the Hartree approximation supports soliton solutions. For the case of the repulsive interactions normally encountered in BEC experiments, the interesting solutions are dark solitons, that is, 'dips' in the density profile of the condensate. These dark solitons have been recently demonstrated in two experiments [4, 7] which appear to be in good agreement with the predictions of the Gross–Pitaevskii equation (GPE).

While very interesting from a fundamental physics point of view, dark solitons would appear to be of limited potential in applications such as atom interferometry, where it is desirable to achieve dispersionless transport of a spatially localized ensemble of atoms, rather than a 'hole'. In that case, bright solitons are much more interesting. However, the problem is that large condensates are necessarily associated with repulsive interactions, for which bright solitons might seem impossible since the nonlinearity cannot compensate for the kinetic energy part (diffraction) in the atomic dynamics. While this is true for atoms in free space, this is however not the case for atoms in suitable potentials, e.g. in optical lattices, and external fields, such as magnetic and laser fields. This is because in that case, it is possible to tailor the dispersion relation of the atoms in such a way that their effective mass becomes negative. For such negative masses, a repulsive interaction is precisely what is required to achieve soliton solutions. This result is known from nonlinear optics, where such soliton solutions, called gap solitons, have been predicted and demonstrated [8–13].

The paper is organized as follows. In section 2 we briefly reduce the three-dimensional (3D) model to a quasi-one-dimensional (quasi-1D) model on the basis of the experimental conditions. This leads to the quasi-1D Gross–Pitaevskii equation, the nonlinear mean-field equation of motion for the atomic condensate. In section 3 we introduce the bright and dark soliton solutions of these equations. In section 4 we demonstrate how to launch and control these solitons. Finally, section 5 gives a summary and conclusion.

2. The model and Hamiltonian

We investigate the dynamics of a condensate under a weakly uniform magnetic field with a time-independent axial component B_1 (which plays a similar role to the intersite transfer that couples the neighbouring states in the electron lattice system) and a time-dependent longitudinal component $B_2 \sin(\omega t)$ (which is analogous to a time-dependent electric field in the electron–lattice system) [14].

Generally speaking, a BEC is essentially a one-dimensional system where, in particular, the lateral motion can be either neglected [15] or confined [4]. This may be realized, for example, using a red-detuned optical dipole trap around the focus of a Gaussian laser beam centred on the origin [16], thereby giving approximately harmonic axial confinement. Thus, the BEC becomes essentially one-dimensional with a cross-sectional area axial to the unbound longitudinal axis. The nonlinear elementary excitations of Bose–Einstein condensates with an external time-independent axial field and a time-dependent longitudinal magnetic field can be described by the Gross–Pitaevskii equation (or nonlinear Schrödinger equation)

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{4\pi\hbar^2}{m}a|\Psi|^2\Psi + [B_1 x + B_2 x\sin(\omega t)]\Psi, \tag{1}$$

where m is the atomic mass, a is the scattering length between atoms, B_1 and B_2 are the time-independent axial and time-dependent longitudinal magnetic field, respectively. Making a dimensionless transformation, we can rewrite equation (1) as

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + g|\psi|^2\psi + x[b + c\sin(\omega t)]\psi, \tag{2}$$

where x is measured in units of $l_0 = 1\ \mu$m, a characteristic length unit in this type of experiment, t in units of $ml_0^2/\hbar$ (m is the atomic mass), ψ in units of the square root of $n_0 a^3$, the maximum density in the initial distribution of the condensate, and the interaction constant is defined as $g = 4\pi n_0 a l_0^2$, with a being the interatomic scattering length.

In this paper, we look for one-soliton and multisoliton solutions of the Gross–Pitaevskii equation (2) under an external time-independent axial and a time-dependent longitudinal magnetic field.

3. The exact soliton solutions of the 1D GPE

The exact solution of the Gross–Pitaevskii equation (GPE) or the nonlinear Schrödinger equation can be achieved by the inverse scattering method [17–19]. Its central step is solving an auxiliary linear scattering problem [6]. In order to use the inverse scattering method, we first introduce the following 'Lax pair':

$$\Phi_x = L\Phi, \qquad \Phi_t = M\Phi, \tag{3}$$

where

$$L = \begin{pmatrix} -\mathrm{i}\lambda(t) & \psi \\ g\psi^* & \mathrm{i}\lambda(t) \end{pmatrix}, \qquad M = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{4}$$

and

$$\begin{aligned} A &= -\mathrm{i}\lambda^2 - \mathrm{i}\alpha x - \mathrm{i}g|\psi|^2/2, \\ B &= \lambda\psi + \mathrm{i}\psi_x/2, \\ C &= g\lambda\psi^* - \mathrm{i}g\psi_x^*/2, \\ \lambda_t &= \alpha = [b + c\sin(\omega t)]/2, \\ \lambda(t) &= \left[bt - \frac{c}{\omega}\cos(\omega t)\right]\Big/ 2 + \zeta + \mathrm{i}\eta, \end{aligned} \tag{5}$$

which can be tested via the compatibility condition $\Phi_{xt} = \Phi_{tx}$. Using the above transformation, we can reduce the Gross–Pitaevskii equation (2) to an auxiliary linear scattering problem (3), and also look for two sets of eigenfunctions (Φ and Ψ) with boundary conditions as follows:

$$\begin{aligned} \Phi &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varsigma(t)\exp(-\mathrm{i}\lambda x) \qquad \text{as } x \to -\infty, \\ \overline{\Phi} &\sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} \varsigma(t)\exp(\mathrm{i}\lambda x) \qquad \text{as } x \to -\infty, \end{aligned} \tag{6}$$

$$\begin{aligned} \Psi &\sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varsigma(t)\exp(\mathrm{i}\lambda x) \qquad \text{as } x \to +\infty, \\ \overline{\Psi} &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varsigma(t)\exp(-\mathrm{i}\lambda x) \qquad \text{as } x \to +\infty, \end{aligned} \tag{7}$$

where $\varsigma(t)$ can be obtained by applying the asymptotic behaviour of the second of equations (3). Suppose that

$$\begin{aligned} \Phi &= r(\lambda, t)\overline{\Psi} + s(\lambda, t)\Psi, \\ \overline{\Phi} &= -\bar{r}(\lambda, t)\Psi + \bar{s}(\lambda, t)\overline{\Psi}, \end{aligned} \tag{8}$$

where r, $\bar{r}$, s and $\bar{s}$ are scattering coefficients, we can work out the following results:

$$\begin{aligned} r(\lambda(t), t) &= r(\lambda(0), 0), \\ \bar{r}(\lambda(t), t) &= \bar{r}(\lambda(0), 0), \end{aligned} \tag{9}$$

$$\begin{aligned} s(\lambda(t), t) &= s(\lambda(0), 0)\exp(2\mathrm{i}\beta(t)), \\ \bar{s}(\lambda(t), t) &= \bar{s}(\lambda(0), 0)\exp(-2\mathrm{i}\beta^*(t)), \end{aligned} \tag{10}$$

where

$$\begin{aligned}
\beta(t) &= \int \lambda^2\,\mathrm{d}t = \theta(t) + \mathrm{i}\delta(t),\\
\beta^*(t) &= \theta(t) - \mathrm{i}\delta(t),\\
\delta(t) &= 2\zeta\eta t + \frac{b}{2}\eta t^2 - \frac{c}{\omega^2}\eta\sin(\omega t),\\
\theta(t) &= \frac{b^2}{12}t^3 + \frac{b\zeta}{2}t^2 + \left(\frac{c^2}{8\omega^2} + \zeta^2 - \eta^2\right)t\\
&\quad + \frac{c^2}{16\omega^3}\sin(2\omega t) - \frac{c\zeta}{\omega^2}\sin(\omega t) - \frac{bc}{2\omega^2}t\sin(\omega t) - \frac{bc}{2\omega^3}\cos(\omega t).
\end{aligned} \tag{11}$$

The above relations (9)–(10) indicate that r and $\bar{r}$ are independent of t. Furthermore, we can get the corresponding coefficients:

$$\begin{aligned}
c_j(t) &= c_j(0)\exp[2\mathrm{i}\theta_j(t) - 2\delta_j(t)],\\
\overline{c_j}(t) &= c_j^*(0)\exp[-2\mathrm{i}\theta_j(t) - 2\delta_j(t)],
\end{aligned} \tag{12}$$

where θ_j and δ_j ($j = 1, \ldots, N$) are defined in equation (5) in which ζ and η are replaced by ζ_j and η_j respectively. Considering the no reflection case, i.e., $\rho = s/r = 0$, $\overline{\rho} = \bar{s}/\bar{r} = 0$ and $\varepsilon = -1$, we can find the following solution of equation (2):

$$\psi(x,t) = -2K_1(x,x,t), \tag{13}$$

by solving the Gelfand–Levitan–Marchenko equations

$$\begin{aligned}
K(x,w,t) &= \begin{pmatrix}1\\0\end{pmatrix}\overline{F}(x+w,t) + \int_x^\infty \overline{K_1}(x,z,t)\overline{F}(z+w,t)\,\mathrm{d}z,\\
\overline{K}(x,w,t) &= -\begin{pmatrix}0\\1\end{pmatrix}F(x+w,t) - \int_x^\infty K_1(x,z,t)F(z+w,t)\,\mathrm{d}z,
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
K(x,w,t) &= \begin{pmatrix}k_1(x,w,t)\\k_2(x,w,t)\end{pmatrix} = \sum_{j=1}^N \begin{pmatrix}k_{1j}(x,t)\\k_{2j}(x,t)\end{pmatrix}\exp(-\mathrm{i}\lambda_j^* w),\\
\overline{K}(x,w,t) &= \begin{pmatrix}\overline{k_1}(x,w,t)\\\overline{k_2}(x,w,t)\end{pmatrix} = \sum_{j=1}^N \begin{pmatrix}\overline{k_{1j}}(x,t)\\\overline{k_{2j}}(x,t)\end{pmatrix}\exp(\mathrm{i}\lambda_j w),
\end{aligned} \tag{15}$$

$$\begin{aligned}
F(x+w,t) &= -\mathrm{i}\sum_{j=1}^N c_j(0)\exp[\mathrm{i}2\theta_j + \mathrm{i}\lambda_j(x+w) - 2\delta_j],\\
\overline{F}(x+w,t) &= \mathrm{i}\sum_{j=1}^N c_j^*(0)\exp[-\mathrm{i}2\theta_j - \mathrm{i}\lambda_j^*(x+w) - 2\delta_j].
\end{aligned} \tag{16}$$

$k_{1j}, k_{2j}, \overline{k}_{1j}, \overline{k}_{2j}$ are functions to be determined, $\lambda_j = (bt - c\cos(\omega t)/\omega)/2 + \vartheta_j$ and the constants $\vartheta_j = \zeta_j + \mathrm{i}\eta_j$. Substituting equations (15)–(16) into equation (14), we can obtain

$$\begin{aligned}
&k_{1l}\exp(-\mathrm{i}\lambda_l^* x) - \mathrm{i}c_l^*\exp(-2\mathrm{i}\lambda_l^* x)\exp(-2\mathrm{i}\theta_j - 2l\delta_l)\\
&\qquad + \overline{k}_{1l}\sum_{j=1}^N \frac{c_j^*}{\lambda_l - \lambda_j^*}\exp(-2\mathrm{i}\theta_j - 2\delta_j)\exp(-\mathrm{i}\lambda_j^* x)\exp[\mathrm{i}(\lambda_l - \lambda_j^*)x] = 0,\\
&\overline{k}_{1l} = -k_{1l}\exp(-\mathrm{i}\lambda_l x)\sum_{j=1}^N \frac{c_j}{\lambda_l - \lambda_l^*}\exp(2\mathrm{i}\theta_j - 2\delta_j)\exp(\mathrm{i}\lambda_j^* x)\exp[\mathrm{i}(\lambda_j - \lambda_l^*)x],
\end{aligned} \tag{17}$$

where $l = 1, 2, \ldots, N$. If we determine the function k_{1l} from equation (17), then we can find the N-soliton solution from equations (13) to (16).

3.1. Bright solitons for $g < 0$ (attractive interaction)

For the sake of simplicity and with no loss the generality, we reduce the one-soliton and two-soliton solutions of equation (2). If $N = 1$, then

$$k_{11} = (ic_1^*/\Delta)\exp(-\mathrm{i}\lambda_1^* x)\exp(-2\mathrm{i}\theta - 2\delta), \tag{18}$$

where

$$\Delta = 1 - \frac{|c_1|^2}{(\lambda_1 - \lambda_1^*)^2}\exp[2\mathrm{i}(\lambda_1 - \lambda_1^*)x]\exp(-4\delta). \tag{19}$$

Let $x_0 = \ln(|c_1(0)|/2\eta)$ and $-\mathrm{i}c_1^*(0) = |c_1(0)|\exp(\mathrm{i}\mu)$; the one-soliton solution of equation (2) reads

$$\psi(x,t) = 2\eta\,\mathrm{sech}[2\eta x + 2\delta(t) - x_0]\exp\left[-\mathrm{i}\left(bt - \frac{c}{\omega}\cos(\omega t) + 2\zeta\right)x - 2\mathrm{i}\theta + \mathrm{i}\phi\right], \tag{20}$$

where $\theta(t)$, $\delta(t)$ can be determined by equation (11). If $N = 2$, then

$$\begin{aligned} k_{11}\exp(-\mathrm{i}\lambda_1^* x) &= (\mathrm{i}c_1^*/\Delta_1)\exp(-2\mathrm{i}\lambda_1^* x)\exp(-2\mathrm{i}\theta_1 - 2\delta_1),\\ k_{12}\exp(-\mathrm{i}\lambda_2^* x) &= (\mathrm{i}c_2^*/\Delta_2)\exp(-2\mathrm{i}\lambda_2^* x)\exp(-2\mathrm{i}\theta_2 - 2\delta_2), \end{aligned} \tag{21}$$

where

$$\begin{aligned} \Delta_1 = 1 - \exp[\mathrm{i}(\lambda_1 - \lambda_1^*)x]&\sum_{j=1}^{2}\frac{c_j}{\lambda_j - \lambda_1^*}\exp(2\mathrm{i}\theta_j - 2\delta_j)\exp(\mathrm{i}\lambda_j x)\exp[\mathrm{i}(\lambda_j - \lambda_1^*)x]\\ &\times\sum_{j=1}^{2}\frac{c_j^*}{\lambda_1 - \lambda_j^*}\exp(-2\mathrm{i}\theta_j - 2\delta_j)\exp(-\mathrm{i}\lambda_j^* x)\exp[\mathrm{i}(\lambda_1 - \lambda_j^*)x],\\ \Delta_2 = 1 - \exp[\mathrm{i}(\lambda_2^* - \lambda_2)x]&\sum_{j=1}^{2}\frac{c_j}{\lambda_j - \lambda_2^*}\exp(2\mathrm{i}\theta_j - 2\delta_j)\exp(\mathrm{i}\lambda_j x)\exp[\mathrm{i}(\lambda_j - \lambda_2^*)x]\\ &\times\sum_{j=1}^{2}\frac{c_j^*}{\lambda_2 - \lambda_j^*}\exp(-2\mathrm{i}\theta_j - 2\delta_j)\exp(-\mathrm{i}\lambda_j^* x)\exp[\mathrm{i}(\lambda_2 - \lambda_j^*)x]. \end{aligned} \tag{22}$$

Using equation (5), we can simplify the above equation (22) as follows:

$$\begin{aligned} \Delta_1 = 1 + \frac{|c_1|^2}{4\eta_1^2}\exp(-4\eta_1 x - 4\delta_1) &+ \frac{|c_2|^2}{(\zeta_1 - \zeta_2)^2 + (\eta_1 + \eta_2)^2}\exp(-4\eta_2 x - 4\delta_2)\\ &- \frac{|c_1 c_2|[(\zeta_1 - \zeta_2)\sin(\wedge) - (\eta_1 + \eta_2)\cos(\wedge)]}{\eta_1[(\zeta_1 - \zeta_2)^2 + (\eta_1 + \eta_2)^2]}\exp[-2(\eta_1 + \eta_2)x - 2(\delta_1 + \delta_2)],\\ \Delta_2 = 1 + \frac{|c_2|^2}{4\eta_2^2}\exp(-4\eta_2 x - 4\delta_2) &+ \frac{|c_1|^2}{(\zeta_1 - \zeta_2)^2 + (\eta_1 + \eta_2)^2}\exp(-4\eta_1 x - 4\delta_1)\\ &- \frac{|c_1 c_2|[(\zeta_1 - \zeta_2)\sin(\wedge) - (\eta_1 + \eta_2)\cos(\wedge)]}{\eta_2[(\zeta_1 - \zeta_2)^2 + (\eta_1 + \eta_2)^2]}\exp[-2(\eta_1 + \eta_2)x - 2(\delta_1 + \delta_2)], \end{aligned} \tag{23}$$

where $\wedge = 2(\zeta_1 - \zeta_2)x + 2(\theta_1 - \theta_2) + (\phi_1 - \phi_2)$. Finally, we obtain the two-soliton solution of equation (2):

$$\psi(x,t) = -2\sum_{j=1}^{2}(\mathrm{i}c_j^*/\Delta_j)\exp(-2\mathrm{i}\lambda_j^* x - 2\mathrm{i}\theta_j - 2\mathrm{i}\delta_j), \tag{24}$$

where Δ_1, Δ_2 can be determined by equation (23). Mathematically, we can find the N-soliton solutions of equation (2) with the help of equations (3) and (11)–(16). The solution (20) is usually referred to as the bright soliton of equation (2). From the viewpoint of the physics, the bright soliton solution is reasonable since the kinetic energy of the BEC can be balanced by the nonlinearity, yielding non-spreading wavepackets for the case of attractive interactions (i.e. $a < 0$). These solutions correspond to spatially localized bright solitons [20, 21].

Such 1D bright solitons could in principle be realized in cigar shaped Bose–Einstein condensates with negative scattering lengths, for example, in ^{7}Li [3] or ^{85}Rb, by using Feshbach resonances [22–24] to tune the scattering length. In two or more dimensions, negative scattering lengths can lead to catastrophic collapse in homogeneous systems for large enough particle numbers. However, in quasi-1D systems with strong axial confinement the solitons are rendered stable [25].

3.2. Dark solitons for $g > 0$ (repulsive interaction)

Soliton solutions are non-spreading solutions of equation (2) which preserve their shape under propagation [26]. The kinetic energy term in the GPE tends to spread wavepackets, and the nature of the solitons depends on the sign of the nonlinearity. For the case of repulsive interactions ($a > 0$), both the kinetic energy and the nonlinearity terms in equation (2) tend to broaden the localized wavepackets, so we do not expect localized, or bright soliton, solutions for that case. Only stationary solitons have a vanishing density at their centre, so they are also referred to as black solitons. However, dark solitons describing localized density dips in an otherwise constant background can arise and are given by [4, 7] with analogous solutions being well known in nonlinear fibre optics; see e.g. [21]. In order to find the dark soliton solution of equation (2) when $g > 0$ ($a > 0$), we look for the following form of solution:

$$\psi(x,t) = 2\eta \tanh(2\eta x + 2\delta(t) - x_0) \times \exp(\mathrm{i}\gamma(t)x - \mathrm{i}2\theta(t) - \mathrm{i}\mu), \tag{25}$$

where $\delta(t)$, $\gamma(t)$ and $\theta(t)$ are real functions to be determined. η, x_0 and μ are real constants. Substituting (25) into equation (2) yields

$$\begin{aligned} \gamma_t &= -b - c\sin(\omega t), \\ \delta_t &= -2\eta\gamma, \\ \theta_t &= 2\eta^2 + \frac{\gamma^2}{4}. \end{aligned} \tag{26}$$

Solving equation (26), one gets that

$$\begin{aligned} \gamma(t) &= -bt + c\cos(\omega t)/\omega - 2\zeta, \\ \theta(t) &= \frac{b^2}{12}t^3 - \frac{b\zeta}{2}t^2 + \left(\zeta^2 + \eta^2 + \frac{c^2}{8\omega^2}\right)t \\ &\quad + \frac{c^2}{16\omega^3}\sin(2\omega t) + \frac{c\zeta}{\omega^2}\sin(\omega t) - \frac{bc}{2\omega^2}t\sin(\omega t) - \frac{bc}{2\omega^3}\cos(\omega t), \end{aligned} \tag{27}$$

and $\delta(t)$ is given by equation (9). Hence, we find the dark solution of equation (2) as follows:

$$\psi(x,t) = 2\eta\tanh(2\eta x + 2\delta(t) - x_0)\exp\left(\mathrm{i}x\left(\frac{c}{\omega}\cos(\omega t) - bt - 2\zeta\right) - \mathrm{i}(2\theta + \mu)\right), \tag{28}$$

where $\delta(t)$ and $\theta(t)$ are determined by equations (11) and (27), respectively. Furthermore, by selecting appropriate transformations and applying Hirota's method, we can also get solutions of equation (2) for N dark solitons. Here these are omitted for simplicity.

4. Results and discussion

The soliton solution is the result of the equilibrium of the nonlinear interaction and a kinetic term. If the interaction of atoms is repulsive, then a dark soliton can be found, Otherwise, a bright soliton will emerge. In order to understand the relations of the above mentioned solutions and external fields, we firstly look at the effect of a time-independent axial external field such as gravity on the solitons, then the interplay of the time-independent axial external field and a time-dependent magnetic field, and finally obtain the localization condition for the solitons. In what follows, unless specified otherwise, we will restrict ourselves to the one-soliton solution of the system with $x_0 = 0$.

4.1. *The effect of a time-independent axial external field* ($b \neq 0, c = 0$)

In order to understand the effect of a time-independent axial external field on Bose–Einstein condensates, we firstly turn off the time-dependent longitudinal magnetic field ($c = 0$); thus the dark and bright solitons read

$$|\psi(x,t)|^2 = 4\eta^2 \tanh^2(2\eta x + 4\zeta\eta t + b\eta t^2) \tag{29}$$

and

$$|\psi(x,t)|^2 = 4\eta^2 \operatorname{sech}^2(2\eta x + 4\zeta\eta t + b\eta t^2). \tag{30}$$

The dynamic behaviours of the centres of the solitons satisfy

$$x + 2\zeta t + \frac{b}{2}t^2 = 0 \text{ for bright and dark solitons.} \tag{31}$$

From equation (31) we learn that if $b > 0$ ($\zeta > 0$), then the dark or bright soliton moves away from the initial point ($x = 0, t = 0$) and never comes back to the initial point. If $b < 0$ ($\zeta > 0$), then the soliton comes back the initial point at $t = 4\zeta/|b|$. Generally speaking, a time-independent axial external field can move the solitons.

4.2. *The interplay of the time-independent axial field and a time-dependent longitudinal magnetic field* ($b \neq 0, c \neq 0$)

Turning on both a time-independent axial external field and a time-dependent longitudinal magnetic field, the dark and bright solitons read

$$|\psi(x,t)|^2 = 4\eta^2 \tanh^2\left(2\eta x + 4\zeta\eta t + b\eta t^2 - \frac{2c}{\omega^2}\eta \sin(\omega t)\right) \tag{32}$$

and

$$|\psi(x,t)|^2 = 4\eta^2 \operatorname{sech}^2\left(2\eta x + 4\zeta\eta t + b\eta t^2 - \frac{2c}{\omega^2}\eta \sin(\omega t)\right). \tag{33}$$

For this case, the dynamic behaviours of the centres of the solitons satisfy

$$2x + 4\zeta t + bt^2 - \frac{2c}{\omega^2}\sin(\omega t) = 0.$$

If we want to localize the solitons at the initial position, the following equation should be satisfied:

$$\left(\frac{2\zeta\omega}{c}\right)(\omega t) + \left(\frac{b}{2c}\right)(\omega t)^2 = \sin(\omega t). \tag{34}$$

Figure 1 displays two functions, $y = (2\zeta\omega/c)(\omega t) + (b/2c)(\omega t)^2$ and $y = \sin(\omega t)$. From figure 1, we learn that the crossing points A, B, C, D, E, F satisfy equation (34) and therefore at the times corresponding to A, B, C, D, E, F bright and dark solitons will come back to the initial position. The smaller the values of $(2\zeta\omega)/c$ and $b/2c$, the longer the time for the solitons to come back to the initial position and the longer the lifespan of soliton localization. Briefly, the

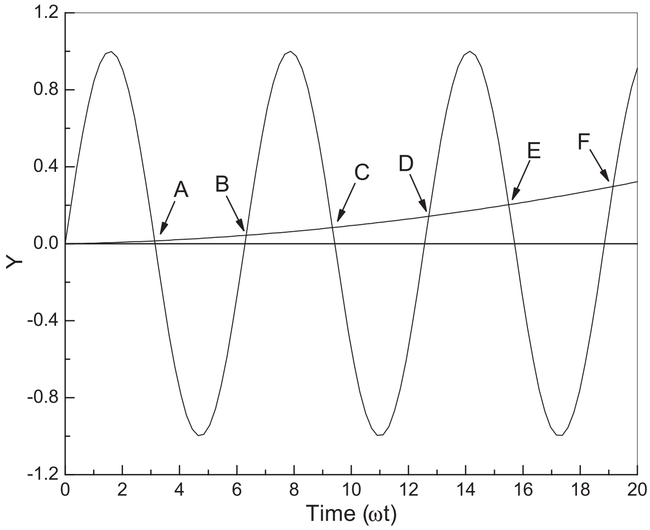

Figure 1. Soliton motion versus gravity and the time-dependent longitudinal magnetic field. The dimensionless parameters are $\zeta = 0.02$ (the initial speed of the soliton), $\omega = 16$, $c = 2048$ (the time-dependent magnetic field), $b = 0.02$ (the time-independent external field), i.e., $2\zeta\omega/c = 0.0027$, $b/2c = 0.000\,67$.

localization of solitons in this case is dependent on the time-dependent longitudinal magnetic field.

For our consideration of the system, we can see that a time-independent axial external field can shift solitons, while time-dependent longitudinal magnetic fields have a tendency to localize solitons at the initial position if the intensity of the magnetic field (c) is very big and the frequency of the magnetic field (ω) is very small, i.e., if it is a slowly oscillating strong magnetic field. By adjusting external parameters, we can move and localize solitons at the initial position for a short time; hence we can perform a lot of experiments based on quantum liquids.

Characterization factors in which experimenters are interested can be obtained from equations (20) and (28). First of all, the group velocity of the soliton solutions is $v = -2\zeta - bt + \frac{c}{\omega}\cos(\omega t)$, which is not a constant but varies as time t. Secondly, the real part ζ of the eigenvalue is responsible for the initial speed of the solitons. The imaginary part η of the eigenvalue determines the height and width of the solitons. Thirdly, the higher the speed at which solitons travel, the higher the peak intensity and the narrower the width of the solitons. Furthermore, the time-dependent wavenumber k and velocity v have the relation $k = v$. Hence, the frequency shift ϖ of the soliton satisfies $\varpi_x = -k_t = b + c\sin(\omega t)$. Finally, both bright (20) and dark (28) soliton solutions are exact solutions of equation (2) and are represented by hyperbolic functions; therefore our exact solutions are stable from the mathematical point of view. This stability can be confirmed by numerical integration of equation (2). Our numerical results, as shown in figure 2, agree well with our exact ones. In addition, the stability of solitons has also been observed by several experimental groups [4, 7, 27, 28]; e.g. see figure 3(b) in [28].

5. Summary

In this paper, we have found exact one-soliton and N-soliton solutions of equation (2) under an external time-independent axial field and a time-dependent longitudinal magnetic field.

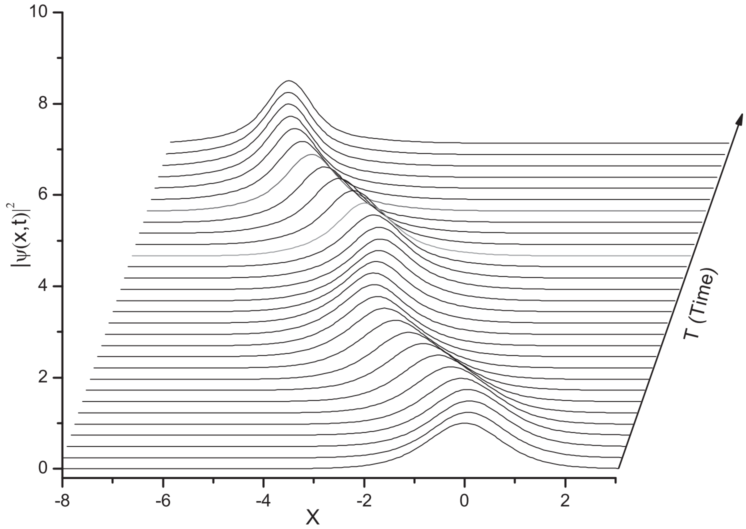

Figure 2. The motion of bright solitons (^{7}Li) versus external fields, where the dimensionless parameters are $b = 0.02$, $c = 150$, $\omega = 6.6$ and $g = -2.0$.

These solitons can be launched in weakly interacting Bose–Einstein condensates. So far, dark and bright solitons have been demonstrated experimentally, and their stabilities have also been observed by several experimental groups [4, 7, 27, 28]. Our results reveal that slowly oscillating strong magnetic fields have some trapping effects for the BEC; the time-independent axial field provides an efficient means of moving solitons of Bose–Einstein condensates, in recent and future experiments [29].

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